ON FUZZY SHORTEST CYCLIC ROUTE PROBLEM

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ABSTRACT

In this paper, the concept of fuzzy shortest cyclic route problem for a salesman is solving by using fuzzy technological values like triangular fuzzy numbers through ranking method. In this proposed ranking method the given fuzzy shortest cyclic route problem is converted into a crisp shortest cyclic route problem and solved by Hungarian. Numerical examples are used to demonstrate the effectiveness and accuracy of this method.

Key Words: Fuzzy numbers, Ranking function, Assignment problem, Hungarian method, Travelling salesman

1. Introduction

In the year of 1965, the concept of fuzzy sets as a new approach for modeling uncertainties was introduced by L.A. Zadeh[14]. This concept provides a natural foundation for treating mathematically the fuzzy phenomena, which exist pervasively in our real world, and for building new branches of fuzzy mathematics. The potential of fuzzy notion was realized by the mathematicians and has successfully been applied in all branches of mathematics. The shortest cyclic route problem was first proposed by W.R. Hamilton in 19^{th} century. In the shortest cyclic route problem for a salesman, there are a number of cities he has to visit. The distance between every pair of cities is known. The salesman is to start from his home city, visit each city only once and return to his home city, and the problem is to identify the shortest route in distance. The shortest cyclic route problem arises in the areas of management like, postal deliveries, inspection, school bus routes, television relays, assembly lines etc. If the distance or time or cost is represented by fuzzy numbers, then the shortest cyclic route problem becomes fuzzy shortest cyclic route problem. In the recent years, several techniques are followed got solving fuzzy shortest cyclic route problem [1, 3,5 -9, 11]. In this paper, the fuzzy shortest cyclic route problem for a salesman is solving by Hungarian method using F.Reuben's ranking function for fuzzy costs as a triangular fuzzy numbers.

2. PRELIMINARIES

In this section, in order to make the exposition self-contained, some basic notions and results used in the sequel are presented.

Definition 2.1: let X be a non-empty set and I the unit interval $[0,1]$. A fuzzy set $\tilde{A}$ of X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.2: The support of a fuzzy set $\tilde{A}$ is the crisp set defined by $\tilde{A} = \{x \in X: \mu_A(x) > 0\}$ and the core of a fuzzy set $\tilde{A}$ is the crisp set defined by $\tilde{A} = \{x \in X: \mu_A(x) = 1\}$ and also the boundary of a fuzzy set defined by $\tilde{A} = \{x \in X: 0 < \mu_A(x) < 1\}$.

Definition 2.3: The α-level set of a fuzzy set $\tilde{A}$ is defined as ordinary set A_α for which the degree of its membership function exceed the α : $A_\alpha = \{x \in X: \mu_A(x) \geq \alpha, 0 \leq \alpha \leq 1\} = [A_\alpha^L, A_\alpha^U]$, where $\begin{cases} A_\alpha^L = \inf\{x \in X: \mu_A(x) \geq \alpha\} \\ A_\alpha^U = \sup\{x \in X: \mu_A(x) \geq \alpha\} \end{cases}$

Definition 2.4: A fuzzy set $\tilde{A}$ in convex set $X = \mathbb{R}^n$ is said to be a convex fuzzy set if and only if its α-level set are convex.

Definition 2.5: A fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ where $a_1, a_2, a_3 \in \mathbb{R}$ is said to be a triangular fuzzy numbers, if its membership function is given by

$$\mu_A(x) = \begin{cases} (x - a_1)/(a_2 - a_1) & a_1 \leq x \leq a_2 \\ x - a_3)/(a_2 - a_3) & a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.6: The interval of confidence for the triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ at α-level set is defined by $A_\alpha = [A_\alpha^L, A_\alpha^U] = [a_1 + (a_2 - a_1)\alpha,\ a_3 - (a_3 - a_2)\alpha]$, for all $\alpha \in [0,1]$.

Definition 2.7: Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. The operations addition, subtraction, multiplication, division and symmetric are defined as follows respectively

(i) $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

(ii) $\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

(iii) $$\tilde{A} \times \tilde{B} = \begin{cases} (a_1b_1,\ a_2b_2, a_3b_3) & a_1 \geq 0 \\ (a_1b_3,\ a_2b_2, a_3b_3) & a_1 < 0, a_3 \geq 0 \\ (a_1b_3,\ a_2b_2, a_3b_1) & a_3 < 0 \end{cases}$$

(iv) $\tilde{A} \div \tilde{B} = (a_1/b_3, a_2/b_2, a_3/b_1)$, where $0 \notin \tilde{B} = (b_1, b_2, b_3)$

(v) if $\tilde{A} = (a_1, a_2, a_3)$ then $-\tilde{A} = (-a_3, -a_2, -a_1)$

Definition 2.8: A ranking function $\Re: F(\mathbb{R}) \rightarrow \mathbb{R}$ where $F(\mathbb{R})$ is the set of fuzzy numbers defined on set of real numbers, maps each fuzzy number into real number, where a natural order exists.

Definition 2.9: The ranking function for a triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ denoted by $\Re(\tilde{A})$ is defined by $\Re(\tilde{A}) = \frac{1}{2}\int_0^1 (a_\alpha^L + a_\alpha^U) d\,\alpha$. Where for every $\alpha \in [0,1]$

$$\begin{cases} a_\alpha^L = a_1 + (a_2 - a_1)\alpha \\ a_\alpha^U = a_3 - (a_3 - a_2)\alpha \end{cases}$$

Theorem 2.10: If $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers, then $\begin{cases} \tilde{A} < \tilde{B} \text{ if and only if } \Re(\tilde{A}) < R(\tilde{B}) \\ \tilde{A} = \tilde{B} \text{ if and only if } \Re(\tilde{A}) = \Re(\tilde{B}) \\ \tilde{A} > \tilde{B} \text{ if and only if } \Re(\tilde{A}) > R(\tilde{B}) \end{cases}$

3. FUZZY SHORTEST CYCLIC ROUTE PROBLEM

There are a number of cities a salesman must visit. The distance (or time or cost) between every pair of cities is known. He starts from his home city, passes through each city once and only once and returns to his home city. The problem is to find the shortest routes in distance (or time or cost). If the salesman is to visit only two cities there is no choice. If the number of cities is three (say A, B and C), of which the starting base is A, there are two possible routes A→B→C and A→C→B. For four cities, there are 6 possible routes. In general for n cities there are $(n-1)!$ possible routes. It may be noted that since the salesman has to visit all the n cities, the shortest route will be independent of the selection of the starting city.

3.1. Mathematical formulation of the fuzzy shortest cyclic route problem

Mathematically, the problem stated as follows:

If $\tilde{C}_{ij}$ is the fuzzy distance (or fuzzy time or fuzzy cost) of going from city i to city j and $x_{ij} = 1$, if the salesman goes directly from city i to j and zero otherwise. Then the problem is to find x_{ij} which is

$$minimize\ \tilde{Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{C}_{ij}\, x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1$$

$$\sum_{i=1}^{n} x_{ij} = 1$$

and $x_{ij} = 0$ or 1; i =1, 2, …n; j =1, 2, …,n, with the additional conditions that no city is to be visited twice before the tour of all the cities is completed and that going from city i directly to i is not permitted, which means $\tilde{C}_{ij} = \infty$.

The fuzzy cost matrix is shown in a tabular form as follows

To city

		1	2	3	…	n
From city	1	∞	$\tilde{C}_{12}$	$\tilde{C}_{13}$	…	$\tilde{C}_{1n}$
	2	$\tilde{C}_{21}$	∞	$\tilde{C}_{23}$	…	$\tilde{C}_{2n}$
	3	$\tilde{C}_{31}$	$\tilde{C}_{32}$	∞	…	$\tilde{C}_{3n}$
	…	…	…	…	…	…
	n	$\tilde{C}_{n1}$	$\tilde{C}_{n2}$	$\tilde{C}_{n3}$	…	∞

It may be noted that there must be only one x_{ij} for each value of i and each value of j. The problem is to find n elements, one in each row and in each column so as to minimize the sum total of the associated fuzzy costs shown in the matrix above. The problem is similar to the fuzzy assignment problem with the difference that there is the additional constraint that no city is to be visited again before the tour of all the cities is completed.

The above fuzzy assignment problem can be solved and it can be hoped that the solution will satisfy the additional constraint. If it does not, it can be adjusted by inspection.

3.2. Procedure for solving fuzzy shortest cyclic route problem

The solution procedure given below and it consists of the following steps:

Step 1: Express the given fuzzy shortest cyclic route problem in a tabular form

Step 2: Using ranking function $\Re(\tilde{C}_{ij})$ proposed by F.Reuben's express the fuzzy cost matrix into crisp form.

Step 3: Since $\Re(\tilde{C}_{ij})$ are crisp values, the above problem is obviously the crisp assignment problem, which can be solved by Hungarian method, the optimal solution of the crisp assignment problem which the optimal solution of the fuzzy assignment problem.

Step 4: If the optimal solution obtained in step 3 satisfies the route condition of fuzzy shortest cyclic route problem. Then it gives the optimum solution of required problem. If not, necessary adjustment is to be made assigning the next best solution so as to satisfy the route condition of fuzzy shortest cyclic route problem.

4. Numerical Examples

In this section we provide two examples to illustrate the obtained theoretical results.

Example 4.1. Consider the fuzzy shortest cyclic route problem with the assignment cost matrix $\tilde{C}_{ij}$ is given whose elements are triangular fuzzy numbers as follows in the table

To→ From↓	A	B	C	D
A	∞	(21, 77, 9)	(4, 26, 8)	(28, 56, 20)
B	(10, 62, 30)	∞	(50, 63, 24)	(40, 55, 10)
C	(60, 94, 80)	(22, 44, 18)	∞	(44, 74, 48)
D	(76, 30, 24)	(45, 35, 45)	(25, 58, 3)	∞

Solution:

The problem is to find the optimal shortest cyclic route for a travelling salesman such that he starts from his home city, passes through each city once and only once and returns to his home city.

Using the ranking function to determine the ranking index values of the cost matrix, given by $\Re(\tilde{C}_{ij}) = \frac{1}{2}\int_0^1 (a_\alpha^L + a_\alpha^U)d\,\alpha$, where for every $\alpha \in [0,1]$

$$\begin{cases} a_\alpha^L = a_1 + (a_2 - a_1)\alpha \\ a_\alpha^U = a_3 - (a_3 - a_2)\alpha \end{cases}$$

Therefore we get, $\Re(\tilde{C}_{12}) = 46$; $\Re(\tilde{C}_{13}) = 16$; $\Re(\tilde{C}_{14}) = 40$; $\Re(\tilde{C}_{21}) = 41$; $\Re(\tilde{C}_{23}) = 50$; $\Re(\tilde{C}_{24}) = 40$; $\Re(\tilde{C}_{31}) = 82$; $\Re(\tilde{C}_{32}) = 32$; $\Re(\tilde{C}_{34}) = 60$; $\Re(\tilde{C}_{41}) = 40$; $\Re(\tilde{C}_{42}) = 40$; $\Re(\tilde{C}_{43}) = 36$;

Thus, the crisp form of the given problem is in the following table

To→ From↓	A	B	C	D
A	∞	46	16	40
B	41	∞	50	40
C	82	32	∞	60
D	40	40	36	∞

Now solving the above problem by using Hungarian method, finally we get the following table

To→ From↓	A	B	C	D
A	∞	27	⟨0⟩	21
B	0 ×	∞	13	⟨0⟩
C	49	⟨0⟩	∞	28
D	⟨0⟩	1	0 ×	∞

The above table provides an optimum solution to the fuzzy assignment problem. According to it the salesman should visit cities are given by

$$A \to C; B \to D; C \to B; D \to A. \qquad \text{i.e. } A \to C; C \to B; B \to D; D \to A$$

This solution also satisfies the additional conditions of the fuzzy shortest cyclic route problem. Hence the shortest cyclic route is $A \to C \to B \to D \to A$ at a minimum cost (or distance or time) of$(142, 155, 60)$.

Example 4.2. Solve the following fuzzy shortest cyclic route problem.

To→ From↓	A	B	C	D	E
A	∞	(1, 4, 3)	(3, 6, 9)	(1, 3, 1)	(6, 1, 4)
B	(3, 1, 7)	∞	(5, 7, 1)	(1, 3, 1)	(8, 1, 2)
C	(11, 3, 7)	(2, 7, 4)	∞	(14, 3, 4)	(5, 3, 5)
D	(2, 2, 2)	(1, 2, 3)	(12, 3, 6)	∞	(5, 6, 7)
E	(1, 1, 9)	(2, 3, 4)	(6, 4, 2)	(1, 3, 17)	∞

Solution:

The problem is to find the optimal shortest cyclic route for a travelling salesman such that he starts from his home city, passes through each city once and only once and returns to his home city.

Using the ranking function to determine the ranking index values of the cost matrix, given by $\Re(\tilde{C}_{ij}) = \frac{1}{2}\int_0^1 (a_\alpha^L + a_\alpha^U) d\,\alpha$, where for every $\alpha \in [0,1]$

$$\begin{cases} a_\alpha^L = a_1 + (a_2 - a_1)\alpha \\ a_\alpha^U = a_3 - (a_3 - a_2)\alpha \end{cases}$$

Therefore we get, $\Re(\tilde{C}_{12}) = 3$; $\Re(\tilde{C}_{13}) = 6$; $\Re(\tilde{C}_{14}) = 2$; $\Re(\tilde{C}_{15}) = 3$; $\Re(\tilde{C}_{21}) = 3$;
$\Re(\tilde{C}_{23}) = 5$; $\Re(\tilde{C}_{24}) = 2$; $\Re(\tilde{C}_{25}) = 3$; $\Re(\tilde{C}_{31}) = 6$; $\Re(\tilde{C}_{32}) = 6$;
$\Re(\tilde{C}_{34}) = 6$; $\Re(\tilde{C}_{35}) = 4$; $\Re(\tilde{C}_{41}) = 2$; $\Re(\tilde{C}_{42}) = 2$; $\Re(\tilde{C}_{43}) = 6$;
$\Re(\tilde{C}_{45}) = 6$; $\Re(\tilde{C}_{51}) = 3$; $\Re(\tilde{C}_{52}) = 3$; $\Re(\tilde{C}_{53}) = 4$; $\Re(\tilde{C}_{54}) = 6$;

Thus, the crisp form of the given problem is in the following table

To→ From↓	A	B	C	D	E
A	∞	3	6	2	3
B	3	∞	5	2	3
C	6	5	∞	6	4
D	2	2	6	∞	6
E	3	3	4	6	∞

Now solving the above problem by using Hungarian method, finally we get the following table

To→ From↓	A	B	C	D	E
A	∞	×0	2	⟨0⟩	×0
B	⟨0⟩	∞	1	×0	×0
C	2	1	∞	3	⟨0⟩
D	×0	⟨0⟩	3	∞	4
E	×0	×0	⟨0⟩	4	∞

The above table provides an optimum solution to the fuzzy assignment problem but not to the fuzzy shortest cyclic route problem as it gives $A \to D \to B \to A$ and $C \to E \to C$ as the solution which means that the salesman should go from A to D, from D to B and then come back to A without visiting cities C and E. This violets the additional condition that the salesman is not to visit any city twice before completing his tour of all the cities. Therefore we now try to find the next best solution which satisfies the route condition also. The next minimum non-zero cost matrix is 1. So we try to bring 1 in to the solution, so we start with making an assignment at (2,3) place instead of zero assignment at (2,1) place. The resulting table is shown below

To→ From↓	A	B	C	D	E
A	∞	×0	2	⟨0⟩	×0
B	×0	∞	⟨1⟩	×0	×0
C	2	1	∞	3	⟨0⟩
D	×0	⟨0⟩	3	∞	4
E	⟨0⟩	×0	×0	4	∞

The above table provides an optimum solution to the fuzzy assignment problem. According to it the salesman should visit cities are given by

$A \to D; B \to C; C \to E; D \to B; E \to A.$ i.e. $A \to D; D \to B; B \to C; C \to E; E \to A.$

This solution also satisfies the additional conditions of the fuzzy shortest cyclic route problem. Hence the shortest cyclic route is $A \to D \to B \to C \to E \to A$ at a minimum cost (or distance or time) of$(13, 16, 19)$.

5. Conclusion

In this paper, the fuzzy shortest cyclic route problem costs are considered as imprecise numbers described by triangular fuzzy numbers. Moreover, the fuzzy shortest cyclic route problem has been transformed into crisp shortest cyclic route problem using ranking function for fuzzy costs and solves it by Hungarian method. The result so obtained in example 4.1 and 4.2 are matched with existing technique. This method is effective and easy to understand, can be applied for solving real life problems.

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